Resolvents and Seiberg-Witten representation for Gaussian β-ensemble

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ABSTRACT

The exact free energy of matrix model always obeys the Seiberg-Witten (SW) equations on a complex curve defined by singularities of the quasiclassical resolvent. The role of SW differential is played by the exact one-point resolvent. We show that these properties are preserved in generalization of matrix models to beta-ensembles. However, since the integrability and Harer-Zagier topological recursion are still unavailable for beta-ensembles, we need to rely upon the ordinary AMM/EO recursion to evaluate the first terms of the genus expansion. Consideration in this paper is restricted to the Gaussian model.

1 Introduction

Seiberg-Witten (SW) prepotentials $\mathcal{F}(\vec{a})$ [1, 2, 3, 4] are defined from the peculiar set of implicit equations:

$$\begin{cases} \vec{a} = \oint_{\vec{A}} \Omega \\[2mm] \frac{\partial \mathcal{F}(\vec{a})}{\partial \vec{a}} = \oint_{\vec{B}} \Omega \end{cases} \tag{1}$$

Here Ω is an $(m, 0)$ analytic form (holomorphic, meromorphic or even possessing essential singularities) on a family of $d = 2m$ complex manifolds with a system of conjugated $(m, 0)$-cycles $\vec{A}$ and $\vec{B}$. When the system (1) is resolvable (its consistency is guaranteed by the Riemann identities) then $\vec{a}$ are called flat coordinates on the moduli space of the family (or simply the flat moduli), and $\mathcal{F}(\vec{a})$ is a "quasiclassical" or Whitham τ-function, on this space, satisfying a set of the (generalized) WDVV equations (usually as a consequence of the "residue formula") [5]. This is by now a classical branch of science, presented in big detail in numerous papers.

A little more recently it has been realized that despite the "quasiclassical" nature of the SW equations, they perfectly survive various "quantization" procedures. The true conceptual meaning of this phenomenon still lacks understanding, but the very fact is getting established more and more reliably. The latest example is the Bohr-Sommerfeld representation [6, 7] of the LMNS free energy [8] in the Nekrasov-Shatshvili limit [9] $\epsilon_2 = 0$: if also $\epsilon_1 = 0$, then this free energy is just the ordinary SW prepotential of [1, 3], but remarkably eqs.(1) survive when at least the first "quantization" parameter ϵ_1 is switched on. Actually it is claimed in [10, 11] that they will survive even further: when both ϵ_1 and ϵ_2 are non-vanishing. And this claim is inspired by the AGT relations [12, 13, 15], which provide a matrix model representation of the LMNS partition function [14, 10]. Then one can use the previous fact of the fundamental importance: that the exact matrix model free energies possess the SW representation with the role of SW differential played by the one-point resolvent

$$\Omega^{MM}(z) = \rho_1(z) = \left\langle \mathrm{Tr} \frac{dz}{z - M} \right\rangle_{MM} \tag{2}$$

which is a meromorphic differential on the spectral curve Σ^{MM}. Again, the SW representation is a kind of straightforward for the planar free energy (where it is actually discussed since [16, 17]) but, remarkably, it survives when all higher-genus corrections in powers of the string coupling constant g_S (i.e. the t'Hooft's coupling $\Lambda = gN$) are switched on. This fact is still less known and under-appreciated. It was mentioned in passing in [18] and in [19], but its real significance can be illustrated by the recent suggestion of [11] to use it in

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a conceptual proof of the AGT relations: by applying the topological recursion procedures [18, 20] to construct a double deformation of the original SW free energy, to $g_s = \sqrt{-\epsilon_1\epsilon_2} \neq 0$ and to $\beta = b^2 = -\epsilon_1/\epsilon_2 \neq 1$.

It is the goal of the present paper to provide more *illustrations* to the SW representation of exact matrix model free energies and to make this crucially important technique more understandable and useable. The paper is dedicated entirely to this issue and we avoid mixing it with the other subjects. The first illustration of this kind was already provided in the Appendix to [11], we reproduce it here and extend to non-unity β. We do not address non-Gaussian models here, since this requires usage of rather heavy techniques, but we will address this question in forthcoming papers. Of course, for AGT applications one needs an essentially non-Gaussian β-ensemble: the open-contour Dotsenko-Fateev integral *a la* [10, 21], which we do not consider in the present text. However, the SW representation undoubtedly exists there as well, also for arbitrary β and in all orders of the genus expansion.

2 The case of $\beta = 1$: a source of questions and educated guesses

The partition function is defined as

$$Z(N) = \frac{1}{N!} \int d\lambda_1 \ldots d\lambda_N \prod_{i<j} |\lambda_i - \lambda_j|^{2\beta} \, e^{-\frac{1}{2g}\sum_i \lambda_i^2} \tag{3}$$

For $\beta = 1$ it is equal to

$$Z(N) = \sqrt{2\pi}^N \sqrt{g}^{N^2} \prod_{k=1}^{N-1} k! \tag{4}$$

Hence, for the free energy $F = \ln Z$ one has (ignoring the terms quadratic and linear in N)

$$F(N) = \sum_{k=1}^{N-1} \ln(k!) \tag{5}$$

It turns out [22] that $Z(N)$ is a Toda-chain τ-function and $F(N)$ possesses the Seiberg-Witten representation (1).

That is, let $\rho_1(z)$ be the one-point resolvent of the model

$$\rho_1(z) = \left\langle \sum_i \frac{1}{z - \lambda_i} \right\rangle \tag{6}$$

Then the system of partial differential SW-equations

$$-\frac{1}{2\pi i} \oint_A \rho_1(z)dz = a \qquad -\oint_B \rho_1(z)dz = \frac{\partial \mathcal{F}_{SW}}{\partial a}, \tag{7}$$

determines the SW prepotential, which, as one can check using the explicit expression for the resolvent from [11], is equal to the free energy

$$\mathcal{F}_{SW}(N) = F(N) \tag{8}$$

and this equality just gives the SW-representation of free energy of the matrix model.

Another remarkable fact is that the one-point resolvent satisfies the difference equation [26, 27]

$$\rho_1(N+1, z) + \rho_1(N-1, z) - 2\rho_1(N, z) = \frac{\partial^2}{\partial z^2}\rho_1(N, z), \tag{9}$$

which implies that its B-periods satisfy [11]

$$\Pi_B(N+1) + \Pi_B(N-1) - 2\Pi_B(N) = -\frac{1}{N} \tag{10}$$

Eq.(9) is closely related to integrability of $Z(N)$, that is, to the Toda chain equation [22]

$$Z(N)\partial_1^2 Z(N) - \partial_1 Z(N)\partial_1 Z(N) = Z(N+1)Z(N-1), \tag{11}$$

where $\partial_1 Z(N) = \langle \sum_i \lambda_i \rangle$ and $\partial_1^2 Z(N) = \left\langle (\sum_i \lambda_i)^2 \right\rangle$.

Eq.(10) is a weaker corollary of (9).

Knowing these facts, the following questions arise naturally:

- Does (8) hold as well in the $\beta \neq 1$ case?

- Is there some β-deformed version of (9) and (10)?

The rest of the paper is devoted to the affirmative answer to the first question. A partial progress in answering the second one is outlined in the Appendix.

3   Resolvents

3.1   Ward identities: generalities

A powerful technique for evaluating correlators in matrix models is known under different names: of the Virasoro constraints, of the loop equations, of the Ward identities [23, 18]. It relies on "the general covariance" of partition function: that is, the invariance of integral under arbitrary change of integration variables. For the eigenvalue model, not obligatory Gaussian, the Virasoro constraints can be deduced as follows [24]. Consider the obvious identity

$$\sum_k \int d\lambda_1 \dots d\lambda_N \frac{\partial}{\partial \lambda_k} \left(\lambda_k^n \Delta^{2\beta} e^{-\frac{1}{g} \sum_i V(\lambda_i)} S_{i_1} \dots S_{i_m} \right) = 0, \tag{12}$$

where $S_i = \sum_a \lambda_a^i$ and Δ is the absolute value of the Van-der-Monde determinant. Here $V(\lambda) = \sum_k T_k \lambda^k$; in the Gaussian case only $T_2 = 1/2$ is non-vanishing.

One can easily check that

$$\sum_k \frac{\partial}{\partial \lambda_k} \left(\lambda_k^n \Delta^{2\beta} \right) = \left(\beta \sum_{a=0}^{n-1} S_a S_{n-1-a} + (1-\beta) n S_{n-1} \right) \Delta^{2\beta} \tag{13}$$

and this is the only piece of equation that changes when one changes β.

Differentiation of the potential term gives

$$\sum_k \lambda_k^n \frac{\partial}{\partial \lambda_k} \left(e^{-\frac{1}{g} \sum_i V(\lambda_i)} \right) = \left(-\frac{1}{g} \sum_a V'(\lambda_a) \lambda_a^n \right) e^{-\frac{1}{g} \sum_i V(\lambda_i)} \tag{14}$$

and this is the only model-dependent part of our consideration.

Differentiation of the remaining terms gives

$$\sum_k \lambda_k^n \frac{\partial}{\partial \lambda_k} \left(S_{i_1} \dots S_{i_m} \right) = \sum_{j=1}^{m} i_j S_{i_1} \dots S_{i_j+n-1} \dots S_{i_m} \tag{15}$$

Now, having all the ingredients of the equations, one can write them in various forms.

Virasoro constraints.   If one denotes the disconnected correlator as

$$C_{i_0,..,i_m} = \langle S_{i_0} \dots S_{i_m} \rangle \tag{16}$$

the above considerations imply that

$$\beta \sum_{a=0}^{n-1} C_{a,n-1-a,i_1,..,i_m} + (1-\beta) n C_{n-1,i_1,..,i_m} - \frac{1}{g} \sum_k k T_k C_{n-1+k,i_1,..,i_m} + \sum_{j=1}^{m} i_j C_{i_1,..,i_j+n-1,..,i_m} = 0 \tag{17}$$

Differential $(\widetilde{W})$ operators.   If one works with the generic partition function (with infinitely many non-fixed times) one can write these equations as a differential equation on the (full) partition function. Namely, let the potential have the form

$$V(\lambda) = (T_0 + t_0) N + \sum_{k=1}^{\infty} (T_k + t_k) \lambda^n, \tag{18}$$

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where T_k are background values of source fields (usually, only finitely many of them are non-zero) and t_k are perturbations of these background values. The partition function is thought of as a formal series in t_k. Note that, for the non-normalized average, one has

$$\langle S_a \rangle = -g \frac{\partial}{\partial t_a} \langle 1 \rangle \tag{19}$$

and, hence, the Virasoro constraints (17) can be written as

$$\frac{\partial}{\partial t_{i_1}} \cdots \frac{\partial}{\partial t_{i_m}} \left(\sum_{k=0}^{\infty} k(T_k + t_k) \frac{\partial}{\partial t_{k-1+n}} + g(1-\beta)\, n \frac{\partial}{\partial t_{n-1}} + g^2 \beta \sum_{a=1}^{n-1} \frac{\partial^2}{\partial t_a \partial t_{n-1-a}} \right) Z = 0 \tag{20}$$

Loop equations. They equations arise when one sums up all the Virasoro constraints with the weights $\frac{1}{z^{n+1}}$ and writes the resulting equation in terms of the resolvents. For this purpose, it is convenient to rewrite the Van-der-Monde part of the identity as

$$\sum_k \frac{\partial}{\partial \lambda_k} \left(\lambda_k^n \Delta^{2\beta} \right) = \left(2\beta \sum_{i<j} \frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j} + \sum_a n \lambda_a^{n-1} \right) \Delta^{2\beta} \tag{21}$$

Now summing up all the contributions one gets

$$\beta r\left(z_0, z_0, z_1, ., z_m \right) + (\beta - 1) \frac{\partial}{\partial z_0} r(z_0, z_1, ., z_m) + \sum_{j=1}^{m} \frac{\partial}{\partial z_j} \frac{r(z_1, ., z_m) - r(z_1, ., z_0, ., z_m)}{z_m - z_0} - \tag{22}$$

$$- \frac{1}{g} \sum_{k=0}^{\infty} k T_k z_0^{k-1} r(z_0, z_1, ., z_m) + \frac{1}{g} \sum_{k=0}^{\infty} k T_k \sum_{j=0}^{k-2} z_0^j \frac{1}{2\pi i} \oint_{\infty} dz\, z^{k-2-j} r(z, z_1, ., z_m) = 0,$$

where $r(z_0, ., z_m)$ is the disconnected resolvent

$$r(z_0, ., z_m) = \left\langle \sum_{i_0} \frac{1}{z_0 - \lambda_{i_0}} \cdots \sum_{i_m} \frac{1}{z_m - \lambda_{i_m}} \right\rangle \tag{23}$$

To solve these equations perturbatively in g one has to rewrite the disconnected resolvents in terms of the connected ones. Then, the iteration procedure becomes well-defined: at each step of the procedure one has a system of linear equations for $\rho_{i,j}$, with fixed value of $i + j$. The expansion in powers k of g, as usual, counts contributions of genus $k/2$ Riemann surfaces in string (or topological) expansion. Here $\rho_{i,j}$ stands for the genus j contribution to the i-point connected resolvent.

3.2 Prerequisite: particular correlators

The Ward identities in the form of the Virasoro constraints are very helpful in evaluating individual correlators $C_{i_1 \ldots i_m}$. The advantage of this method is that the answers are exact in g and one may not rewrite the disconnected correlators in terms of the connected ones for the iteration procedure to work (this simplifies the work drastically if one uses symbolic computer computations).

To give an impression of what individual correlators look like we provide the first few one- and two-point correlators. Note that K denotes the *connected* correlators, and $\Lambda \equiv Ng$.

$$
\begin{aligned}
K_k =& & C_k = \langle \sum_i \lambda_i^k \rangle = \langle\langle \sum_i \lambda_i^k \rangle\rangle \hspace{3cm} (24)\\
K_0 =& & \Lambda \\
K_2 =& & \Lambda(\beta\Lambda - \beta + 1) \\
K_4 =& & \Lambda \left(2\beta^2\Lambda^2 - 5\beta^2\Lambda + 3\beta^2 + 5\beta\Lambda - 5\beta + 3 \right) \\
K_6 =& \; 5\beta^3\Lambda^4 + \left(22\beta^2 - 22\beta^3\right)\Lambda^3 + \left(32\beta^3 - 54\beta^2 + 32\beta\right)\Lambda^2 + \left(-15\beta^3 + 32\beta^2 - 32\beta + 15\right)\Lambda \\
K_8 =& \; 14\beta^4\Lambda^5 + \left(93\beta^3 - 93\beta^4\right)\Lambda^4 + \left(234\beta^4 - 398\beta^3 + 234\beta^2\right)\Lambda^3 \\
& + \left(-260\beta^4 + 565\beta^3 - 565\beta^2 + 260\beta\right)\Lambda^2 + \left(105\beta^4 - 260\beta^3 + 331\beta^2 - 260\beta + 105\right)\Lambda
\end{aligned}
$$

$$\cdots$$

$$K_{k,j} = \qquad C_{k,j} - C_k C_j = \left\langle\left\langle \sum_i \sum_l \lambda_i^k \lambda_l^j \right\rangle\right\rangle \qquad (25)$$

$$K_{1,1} = \qquad \Lambda$$

$$K_{1,3} = \qquad 3\Lambda(\beta(\Lambda - 1) + 1)$$
$$K_{2,2} = \qquad 2\Lambda(\beta(\Lambda - 1) + 1)$$

$$K_{1,5} = \quad 10\beta^2\Lambda^3 + 5\left(5\beta - 5\beta^2\right)\Lambda^2 + 5\left(3\beta^2 - 5\beta + 3\right)\Lambda$$
$$K_{2,4} = \qquad 4\Lambda(\beta(\Lambda - 1)(\beta(2\Lambda - 3) + 5) + 3)$$
$$K_{3,3} = \qquad 3\Lambda(\beta(\Lambda - 1)(\beta(4\Lambda - 5) + 9) + 5)$$

$$\cdots$$

In terms of **the CFT-inspired variables** $M = b\Lambda$ and $Q = b - \frac{1}{b}$, $b = \sqrt{\beta}$ these read

$$K_0 = \qquad \frac{M}{b} \qquad (26)$$
$$K_2 = \qquad M(M - Q)$$
$$K_4 = \qquad Mb(1 + 2M^2 - 5MQ + 3Q^2)$$
$$K_6 = \qquad Mb^2(5M(2 + M^2) - (13 + 22M^2)Q + 32MQ^2 - 15Q^3)$$
$$K_8 = \quad Mb^3(21 + 14M^4 - 93M^3Q + 160Q^2 + 105Q^4 - 5MQ(43 + 52Q^2) + M^2(70 + 234Q^2))$$

$$\cdots$$

$$K_{1,1} = \qquad \frac{M}{b} \qquad (27)$$

$$K_{1,3} = \qquad 3M(M - Q)$$
$$K_{2,2} = \qquad 2M(M - Q)$$

$$K_{1,5} = \quad 5Mb(1 + 2M^2 - 5MQ + 3Q^2)$$
$$K_{2,4} = \quad 4Mb(1 + 2M^2 - 5MQ + 3Q^2)$$
$$K_{3,3} = \quad 3Mb(1 + 4M^2 - 9MQ + 5Q^2)$$

$$\cdots$$

Note the remarkable simplification in comparison with (24) and (25).

3.3 The answer for resolvent at $\beta = 1$

Just for completeness (and in part to emphasize the relative complexity of the $\beta \neq 1$ case) we begin from the well-known one-point resolvent at $\beta = 1$ [18]:

$$\rho_1 = \left\langle \sum_i \frac{1}{z - \lambda_i} \right\rangle = \sum_{k=0}^{\infty} \rho_{1,k} g^k \qquad (28)$$

The particular genus contributions are

$$\rho_{1,0}(z) = \frac{1}{2}(z - y(z))$$

$$\rho_{1,2}(z) = \frac{\Lambda}{y(z)^5} \qquad (29)$$

$$\rho_{1,4}(z) = \frac{21\Lambda(\Lambda + z^2)}{y(z)^{11}}$$

5

$$\rho_{1,6}(z) = \frac{11\Lambda(158\Lambda^2 + 558\Lambda z^2 + 135 z^4)}{y(z)^{17}},$$

$$\cdots \tag{30}$$

where $y(z)^2 = z^2 - 4\Lambda$ and all $\rho_{1,2k+1}$ vanish. General formulae for $\rho_{1,2n}$ can be obtained by integral transformation from exact Harer-Zagier functions, see [26, 18, 27].

3.4 The answer for ρ_1 at generic $\beta \neq 1$

The loop equations (22) in the case of Gaussian model acquire a very simple form

$$\beta r(z_0, z_0, z_1, ., z_m) + (\beta - 1)\frac{\partial}{\partial z_0} r(z_0, z_1, ., z_m) + \sum_j \frac{\partial}{\partial z_j} \frac{r(z_1, ., z_j, ., z_m) - r(z_1, ., z_0, ., z_m)}{z_j - z_0} - \tag{31}$$

$$-\frac{1}{g} z_0 r(z_0, z_1, ., z_m) + \frac{\Lambda}{g^2} r(z_1, ., z_m) = 0$$

where r denotes the disconnected resolvent

$$r(z_1, ., z_m) = \left\langle \sum_{i_1} \frac{1}{z_1 - \lambda_{i_1}} \cdots \sum_{i_m} \frac{1}{z_m - \lambda_{i_m}} \right\rangle \tag{32}$$

To solve this system of equations one should rewrite the disconnected correlators in terms of the connected ones and substitute the connected correlators by their Laurent expansion [25].

Thus, assuming that $\rho_1(z) = \frac{1}{g} \sum_{i=0}^{\infty} \rho_{1,i}(z) \cdot g^i$ (so that the even parts of ρ are associated with oriented surfaces, while the odd parts with the non-oriented ones, with half-integer genera), one gets for the first few terms:

$$\rho_{1,0}(z) = \frac{z}{2\beta} - \frac{y(z)}{2\beta} = \frac{1}{2\beta}(z - y(z))$$

$$\rho_{1,1}(z) = \frac{\frac{1}{2} - \frac{1}{2\beta}}{y(z)} + \frac{\frac{z}{2\beta} - \frac{z}{2}}{y(z)^2} = \frac{\beta - 1}{2\beta y(z)}\left(1 - \frac{z}{y(z)}\right)$$

$$\rho_{1,2}(z) = \frac{5\beta^2 \Lambda - 9\beta\Lambda + 5\Lambda}{y(z)^5} + \frac{\beta + \frac{1}{\beta} - 2}{y(z)^3} + \frac{-\beta z - \frac{z}{\beta} + 2z}{y(z)^4} \tag{33}$$

$$\rho_{1,3}(z) = (\beta - 1)\left(\frac{10 - 19\beta + 10\beta^2}{2\beta y(z)^5}\left(1 - \frac{z}{y(z)}\right) + \frac{5\Lambda(5 - 9\beta + 5\beta^2)}{y(z)^7} + \frac{-\Lambda z(30 - 43\beta + 30\beta^2)}{y(z)^8}\right)$$

$$\rho_{1,4}(z) = \frac{37\beta^3 - \frac{273\beta^2}{2} + 199\beta + \frac{37}{\beta} - \frac{273}{2}}{y(z)^7} + \frac{-37\beta^3 z + \frac{273\beta^2 z}{2} - 199\beta z - \frac{37z}{\beta} + \frac{273z}{2}}{y(z)^8} + \frac{419\beta^4 \Lambda - 1357\beta^3 \Lambda + 1897\beta^2 \Lambda - 1357\beta\Lambda + 419\Lambda}{y(z)^9}$$
$$+ \frac{-240\beta^4 \Lambda z + 824\beta^3 \Lambda z - 1168\beta^2 \Lambda z + 824\beta\Lambda z - 240\Lambda z}{y(z)^{10}} + \frac{1105\beta^5 \Lambda^2 - 3240\beta^4 \Lambda^2 + 4375\beta^3 \Lambda^2 - 3240\beta^2 \Lambda^2 + 1105\beta\Lambda^2}{y(z)^{11}}$$

$$\rho_{1,5}(z) = (\beta - 1)\left[\frac{(706 - 2379\beta + 3367\beta^2 - 2379\beta^3 + 706\beta^4)}{2\beta y^9(z)\left(1 - \frac{z}{y(z)}\right)} + \frac{4351 - 13458\beta + 18508\beta^2 - 13458\beta^3 + 4351\beta^4}{y(z)^{11}}\right.$$
$$- \frac{3\Lambda z(1530 - 4241\beta + 5764\beta^2 - 4241\beta^3 + 1530\beta^4)}{y(z)^{12}} + \frac{55\beta\Lambda^2(221 - 648\beta + 875\beta^2 - 648\beta^3 + 221\beta^4)}{y(z)^{13}}$$
$$\left. - \frac{4\beta\Lambda^2 z(3390 - 7883\beta + 10420\beta^2 - 7883\beta^3 + 3390\beta^4)}{y(z)^{14}}\right]$$

$$\rho_{1,6}(z) = \frac{4081\beta^5 - \frac{40405\beta^4}{2} + 44699\beta^3 - 57155\beta^2 + 44699\beta + \frac{4081}{\beta} - \frac{40405}{2}}{y(z)^{11}} + \frac{-4081\beta^5 z + \frac{40405\beta^4 z}{2} - 44699\beta^3 z + 57155\beta^2 z - 44699\beta z - \frac{4081z}{\beta} + \frac{40405z}{2}}{y(z)^{12}}$$
$$+ \frac{77597\beta^6 \Lambda - 340402\beta^5 \Lambda + 702694\beta^4 \Lambda - 878293\beta^3 \Lambda + 702694\beta^2 \Lambda - 340402\beta\Lambda + 77597\Lambda}{y(z)^{13}}$$
$$+ \frac{-59040\beta^6 \Lambda z + 269328\beta^5 \Lambda z - 564000\beta^4 \Lambda z + 707424\beta^3 \Lambda z - 564000\beta^2 \Lambda z + 269328\beta\Lambda z - 59040\Lambda z}{y(z)^{14}}$$
$$+ \frac{451720\beta^7 \Lambda^2 - 1792898\beta^6 \Lambda^2 + 3483419\beta^5 \Lambda^2 - 4266464\beta^4 \Lambda^2 + 3483419\beta^3 \Lambda^2 - 1792898\beta^2 \Lambda^2 + 451720\beta\Lambda^2}{y(z)^{15}}$$
$$+ \frac{-189840\beta^7 \Lambda^2 z + 821128\beta^6 \Lambda^2 z - 1656256\beta^5 \Lambda^2 z + 2049936\beta^4 \Lambda^2 z - 1656256\beta^3 \Lambda^2 z + 821128\beta^2 \Lambda^2 z - 189840\beta\Lambda^2 z}{y(z)^{16}}$$
$$+ \frac{828250\beta^8 \Lambda^3 - 3012930\beta^7 \Lambda^3 + 5531740\beta^6 \Lambda^3 - 6644070\beta^5 \Lambda^3 + 5531740\beta^4 \Lambda^3 - 3012930\beta^3 \Lambda^3 + 828250\beta^2 \Lambda^3}{y(z)^{17}},$$

here $y\left(z\right)^2 = z^2 - 4\Lambda\beta$ defines the spectral curve, which in this case is the torus with a degenerated handle (located at infinity of the complex plane).

$\beta \to \frac{1}{\beta}$ **symmetry.** The AGT relation implies that the β-deformed matrix model should be related to some CFT, with the central charge of the corresponding CFT given by

$$c = 1 - 6\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)^2 \tag{34}$$

This hints that there should be the symmetry $\beta \to \frac{1}{\beta}$ present in the matrix model despite this is far from obvious in the original expression (3). And, indeed, one can see that if one rescales the quantities in the following way:

$$z' = \sqrt{\beta}z, \qquad \rho'_{1,g} = \sqrt{\beta}^{g+1}\rho_{1,g}, \tag{35}$$

the resulting $\rho'_{1,g}$ are symmetric w.r.t. $\beta \to \frac{1}{\beta}$.

4 Seiberg-Witten construction

4.1 Ideology

The Seiberg-Witten construction, originally proposed to obtain the low-energy effective action in $\mathcal{N} = 2$ SUSY gauge theory is in fact a manifestation of a more general statement.

The starting objects in the SW representation are the algebraic curve and the meromorphic differential λ_{SW} on it. Given such a data, one writes the following system of equations

$$\oint_{A_i} \lambda_{SW} \sim a_i \qquad \oint_{B_i} \lambda_{SW} \sim \frac{\partial \mathcal{F}_{SW}}{\partial a_i}, \tag{36}$$

where A_i and B_i form a symplectic basis of cycles on the algebraic curve and proportionality coefficients in equations slightly depends on setting.

It turns out that a huge source of the SW data is provided by the eigenvalue models (EVM). Namely, the algebraic curve is the spectral curve of the given EVM, while the SW differential is $\rho_1(z)dz$, ρ_1 being the one-point resolvent. Note that the original SW construction corresponds to the zeroth order of genus expansion of the resolvent in g, and taking into account further terms of the expansion corresponds to the deformation (quantization) of the original SW differential and prepotential. Remarkably, the *all genus* free energy, not only its genus zero part, continues to satisfy the SW equations.

We fix the proportionality coefficients in the SW equations as follows

$$-\frac{1}{2\pi i}\oint_{A_i} \rho_1(z)dz = a_i \qquad -\beta\oint_{B_i} \rho_1(z)dz = \frac{\partial \mathcal{F}_{SW}}{\partial a_i}, \tag{37}$$

as relation between the free energy and the SW prepotential is most transparent in this way. Note that β appeared as a coefficient in the second equation, [7].

4.2 Calculation of A- and B-periods

Now let us apply the SW construction to ρ_1 that we found in section 3.4.

The spectral curve is given by the equation

$$y^2 = z^2 - 4\Lambda\beta \tag{38}$$

The A-cycle encircles the ramification points $-\sqrt{4\Lambda\beta}$ and $\sqrt{4\Lambda\beta}$, while the B-cycle encircles $\sqrt{4\Lambda\beta}$ and ∞.

Since in this case the value of A-period is equal to the residue at infinity, the A-period gets contributions only from $\rho_{1,0}$ and $\rho_{1,1}$:

$$a = -\frac{1}{2\pi i}\oint_{-\sqrt{4\Lambda\beta}}^{\sqrt{4\Lambda\beta}} \rho(z)dz = N + \frac{1-\beta}{2\beta} \tag{39}$$

Note at this point that the dependence $a(N)$ is *linear* and one can safely substitute $\frac{\partial}{\partial a}$ by $\frac{\partial}{\partial N}$ in the SW equation to simplify calculations.

Evaluating the B-periods is more tricky, the following formula is of great use

$$\oint_{\sqrt{4\Lambda\beta}}^{+\infty} \frac{dz}{y(z)^p} = \frac{1}{2^{2p-3} (\Lambda\beta)^{(p-1)/2}} \frac{\Gamma(p-1)\Gamma(1-p/2)}{\Gamma(p/2)} \tag{40}$$

To deduce this formula, one has to make the change of variables $z = \frac{2-\zeta}{\zeta}\sqrt{4\Lambda\beta}$ and notice that the resulting integral is proportional to the integral representation for the Euler B-function

$$\oint_{\sqrt{4\beta\Lambda}}^{+\infty} \left(z^2 - 4\beta\Lambda\right)^{-p/2} dz = (4\beta\Lambda)^{-p/2+1/2} \int_1^{+\infty} \left(w^2 - 1\right)^{-p/2} dw = \tag{41}$$

$$= (4\beta\Lambda)^{-p/2+1/2} \, 4^{-p/2} 2 \oint_0^1 (1-\zeta)^{-p/2} \zeta^{p-2} d\zeta = 2^{-2p+3} (\beta\Lambda)^{-p/2+1/2} \frac{\Gamma\left(1 - \frac{p}{2}\right)\Gamma(p-1)}{\Gamma\left(\frac{p}{2}\right)}$$

The terms in (33) with odd powers of z do not contribute to the periods, since they are total derivatives. For instance,

$$\oint_{\sqrt{4\Lambda\beta}}^{+\infty} \frac{zdz}{y(z)^p} = -\frac{1}{p-2} \oint_{\sqrt{4\Lambda\beta}}^{+\infty} d\left(\frac{1}{y(z)^{p-2}}\right) = 0 \tag{42}$$

Note that this is a contour integral and the contour does not pass through the singularities of the integrand.

Alternatively, one may exploit the fact that $y(z)$ satisfies the differential equation

$$\frac{\partial}{\partial\Lambda} y^p = -2\beta p y^{p-2}, \tag{43}$$

and so do its B-periods. Together with the initial conditions

$$\oint_B \frac{dz}{y(z)} = -\ln\beta\Lambda \tag{44}$$

and

$$\oint_B y^p dz = 0\Big|_{\Lambda=0}; \quad p \neq -1, \tag{45}$$

this gives (only minor modifications occur in comparison with $\beta = 1$ case)

n	$\oint_B y^n$	$\oint_B y^{-n}$
1	$-2\beta(\Lambda - \Lambda\log(\beta\Lambda))$	$-\log(\beta\Lambda)$
3	$-6\beta\left(\beta\Lambda^2\log(\beta\Lambda) - \frac{3\beta\Lambda^2}{2}\right)$	$-\frac{1}{2\beta\Lambda}$
5	$-10\beta\left(\frac{11\beta^2\Lambda^3}{3} - 2\beta^2\Lambda^3\log(\beta\Lambda)\right)$	$\frac{1}{12\beta^2\Lambda^2}$
7	$-14\beta\left(5\beta^3\Lambda^4\log(\beta\Lambda) - \frac{125\beta^3\Lambda^4}{12}\right)$	$-\frac{1}{60\beta^3\Lambda^3}$
9	$-18\beta\left(\frac{959\beta^4\Lambda^5}{30} - 14\beta^4\Lambda^5\log(\beta\Lambda)\right)$	$\frac{1}{280\beta^4\Lambda^4}$
11	$-22\beta\left(42\beta^5\Lambda^6\log(\beta\Lambda) - \frac{1029\beta^5\Lambda^6}{10}\right)$	$-\frac{1}{1260\beta^5\Lambda^5}$
13	$-26\beta\left(\frac{11979\beta^6\Lambda^7}{35} - 132\beta^6\Lambda^7\log(\beta\Lambda)\right)$	$\frac{1}{5544\beta^6\Lambda^6}$

Thus, for the B-periods of $\rho_{1,i}$ one gets (in the case of $\rho_{1,0}$ and $\rho_{1,1}$ one has to evaluate the integrals for $p = -1 + \epsilon$ and $1 + \epsilon$ respectively, and then to neglect the terms which diverge as $\epsilon \to 0$; since these terms are constant and linear in Λ, this is safe)

$$\oint_B \rho_{1,0}(z)dz = -\Lambda\ln\Lambda$$

$$\oint_B \rho_{1,1}(z)dz = \frac{1-\beta}{2\beta}\ln\Lambda$$

$$\oint_B \rho_{1,2}(z)dz = \frac{-1+3\beta-\beta^2}{12\beta^2\Lambda} \tag{46}$$

$$\oint_B \rho_{1,3}(z)dz = \frac{1-\beta}{24\beta^2\Lambda^2}$$

$$\oint_B \rho_{1,4}(z)dz = \frac{1 - 5\beta^2 + \beta^4}{360\beta^4\Lambda^3}$$

$$\oint_B \rho_{1,5}(z)dz = \frac{-1 + \beta^3}{240\beta^4\Lambda^4}$$

$$\oint_B \rho_{1,6}(z)dz = \frac{-2 + 7\beta^2 + 7\beta^4 - 2\beta^6}{2520\beta^6\Lambda^5}$$

Remarkably, despite the complexity of $\rho_{1,i}$ grows very fast (exponentially) with increasing i, their B-periods complexity increases not so fast (linearly).

Already at this stage one can see that these formulas agree with the generic ones from [25]

$$\oint_B \rho_{1,2m+2} = \sum_{s=0}^{m+1} B_{2m-2s}B_{2s}\frac{\Gamma(2m+1)}{\Gamma(2s+1)\Gamma(2m-2s+3)}\beta^{-2s}\frac{1}{N^{2m+1}}, \ m \geq 0$$

$$\oint_B \rho_{1,2m+1} = \left(\frac{1}{2\beta} - \frac{1}{2\beta^{2m}}\right)\frac{B_{2m+2}(2m-1)}{(2m+1)(2m+2)}\frac{1}{N^{2m}}, m \geq 1, \tag{47}$$

In [25] they were deduced from eq.(51), which we are now aiming to derive.

4.3   Relation to free energy

Partition function for the β-deformed Gaussian eigenvalue model is defined as

$$Z(N) = \frac{1}{N!}\int d\lambda_1 \ldots d\lambda_N \prod_{i<j}|\lambda_i - \lambda_j|^{2\beta} e^{-\frac{1}{2g}\sum_i \lambda_i^2} \tag{48}$$

and can be calculated explicitly. Generalization of (4) for $\beta \neq 1$ is (see [25])

$$Z(N) = \sqrt{2\pi}^N \sqrt{g}^{\beta N^2 + (1-\beta)N} \prod_{k=1}^{N} \frac{\Gamma(1 + \beta k)}{\Gamma(1 + \beta)} \cdot \frac{1}{\Gamma(N+1)} \tag{49}$$

Now we are ready to check that the free energy

$$F(N) = \ln Z \sim \sum_{k=1}^{N} \ln\Gamma(1 + \beta k) - \ln N!, \tag{50}$$

is equal to the SW prepotential.

Indeed, one can calculate the N-derivative of $F(N)$ and apply the Euler-Maclaurin formula, see eq.(72) in the Appendix, to obtain

$$\frac{\partial}{\partial N}F\left(\frac{\Lambda}{g}\right) = \frac{1}{g}\beta\Lambda\ln\Lambda + \frac{\beta-1}{2}\ln\Lambda + g\frac{1-3\beta+\beta^2}{12\beta\Lambda} + g^2\frac{\beta-1}{24\beta\Lambda^2} + g^3\frac{-1+5\beta^2-\beta^4}{360\beta^3\Lambda^3} +$$

$$g^4\frac{1-\beta^3}{240\beta^3\Lambda^4} + g^5\frac{2-7\beta^2-7\beta^4+2\beta^6}{2520\beta^5\Lambda^5} + o\left(\frac{1}{\Lambda^5}\right)$$

this expression can be now compared with (45), taking into account the factor $-\beta$ in (37).

Finally, one obtains

$$\boxed{F = \mathcal{F}_{SW}} \tag{51}$$

This is the main statement of the paper:

The exact free energy of the Gaussian β-ensemble satisfies the SW equations (37) with the exact resolvents in the role of the SW differential.

Acknowledgements

A   Appendix. Towards understanding of $\beta \neq 1$

In this Appendix we outline a few topics which are poorly understood but are of crucial importance for the future theory of β-ensembles.

A.1   Integrability

At $\beta = 1$ we saw that the free energy and the resolvent satisfy integrable differential-difference equations (9) and (10). These equations are intimately related with the Toda integrable structure of the Gaussian matrix model (in general case, it becomes KP integrability). In particular, the Toda equation can be written as follows

$$\frac{\partial^2}{\partial t_1^2} \ln Z(N) = \frac{Z(N+1)Z(N-1)}{Z^2(N)} \tag{52}$$

and, in terms of the free energy,

$$F(N+1) - 2F(N) + F(N-1) = \ln\left(\frac{\partial^2}{\partial t_1^2}F(N)\right) \tag{53}$$

Then, by differentiating w.r.t. t_i and applying the Virasoro constraints, one obtains

$$K_i(N+1) - 2K_i(N) + K_i(N-1) = \frac{i(i-1)}{N}K_{i-2}(N) \tag{54}$$

Summing these equations with the weights $\frac{1}{z^{i+1}}$, one obtains eq.(9). Eq.(10) then follows from (9) if one integrates it along the B-period on the spectral curve.

What is the β-deformation of Toda/KP integrability is a very important and intriguing question, but hard to tackle straightforwardly. As we shall see, eq.(10) can be β-deformed but integrability requires more: a β-deformation of (9) which is still unknown.

A.1.1   Difference equation for periods

For $\beta = 1$ the equation (10) reads

$$\Pi_B(\Lambda + 1) - 2\Pi_B(\Lambda) + \Pi_B(\Lambda - 1) = -\frac{1}{\Lambda}, \tag{55}$$

where $\Pi_B(\Lambda)$ stands for the B period of ρ.

Experimentally one can see (e.g. expanding the l.h.s. into the $\frac{1}{\Lambda}$ series) that, for $\beta \neq 1$, this equation deforms to

$$\Pi_B\left(\Lambda + \frac{1}{\beta}\right) - \Pi_B(\Lambda) - \Pi_B\left(\Lambda + \frac{1-\beta}{\beta}\right) + \Pi_B(\Lambda - 1) = -\frac{1}{\beta\Lambda} \tag{56}$$

A.1.2   Difference equation for resolvents

We, however, were unable to find a β-deformed version of (9), even the corrections of the first order in $\beta - 1$ are missing. What one can say for sure is that in the required generalization the both sides of (9) deform stronger than they do in (56).

A.2 Harer-Zagier topological recursion

A detailed description of the Harer-Zagier functions for $\beta = 1$ can be found e.g. in [27]. Description of matrix model correlators in terms of the resolvents has two advantages: it provides the Ward identities (17) in a simple form of the loop equations (22) and it reveals the important hidden structure, the spectral curve. The drawbacks are the divergency of series for the genus expansion and the lack of explicit formulas for exact correlators.

The last two problems are resolved, e.g., by switching from the exact resolvents to the Harer-Zagier functions, where the correlators are weighted with additional factorial factors, i.e. by summing up the series expansion by the Pade method.

For $\beta \neq 1$ much less is known. So far we were able to obtain the Harer-Zagier functions only for specific values of $\beta \neq 1$. Attempts to evaluate, at least, the first $\beta - 1$ correction lead to some generalizations of the hypergeometric equations which is a hint that something conceptual needs to be done for the results to become simple for arbitrary β. Below our preliminary results are summarized.

The one-point Harer-Zagier generating function is defined as

$$\phi(z) = \frac{4\beta}{\tau^2 - 1} \sum_{k=0}^{\infty} \sum_{N=0}^{\infty} C_k \left(\frac{N}{\beta}, \beta \right) \frac{z^k}{(2k-1)!!} \left(\frac{\tau - 1}{\tau + 1} \right)^N , \tag{57}$$

where $C_k \left(\frac{N}{\beta}, \beta \right)$ is the one-point correlator in the β-deformed matrix model with matrix size equal to N/β.

The case of $\beta = 1$.

$$\phi(\beta = 1, z, \tau) = \frac{1}{1 - \tau z^2} \tag{58}$$

This is the classical result by J.Harer and D.Zagier.

It satisfies the differential equation derived from the integrability conditions

$$\lambda \frac{\partial}{\partial \lambda} \left(\frac{(1-\lambda)^2}{\lambda} \varphi(\lambda, x) \right) = x \frac{\partial}{\partial x} \left(x^2 \varphi(\lambda, x) \right), \tag{59}$$

where $\varphi = \frac{\tau^2 - 1}{4} \phi$ and $\lambda = \frac{\tau - 1}{\tau + 1}$.

Now it turns out that the two- and three-point Harer-Zagier functions can be found as well, and they are expressed through the arctangent function [27], i.e. remain elementary functions.

The case of $\beta = 2$. The Harer-Zagier function for $SO(N)$ matrix model has the form

$$\phi(\beta = 2, z, \tau) = \frac{\tau}{\tau - z - z^2 \tau} + \frac{\sqrt{z}(\tau + 1)}{2} \frac{\arctan \left(\frac{2\sqrt{z}(z\tau - z - 1)\sqrt{\tau - z - z^2 \tau}}{1 + (2 - 3\tau)z + (2 - 2\tau + 2\tau^2)z^2} \right)}{(\tau - z - z^2 \tau)^{3/2}} \tag{60}$$

and it satisfies

$$\left[\frac{2z + 2\tau^2 z + \tau}{2z} + (z - \tau + \tau^2 z) \frac{\partial}{\partial z} \right] \phi(\beta = 2, z, \tau) = \frac{\tau}{2z} + \frac{2 + 2\tau + 2\tau^2 z + \tau^2}{2(2z + 1)(1 - z\tau)} \tag{61}$$

with the initial conditions $\phi(\beta = 2, z, \tau) = 1 + (\tau - 1)z + \ldots$

The case of $\beta = 1/2$. The Harer-Zagier function for the Sp(N) matrix model has the form

$$\phi(\beta = 1/2, z, \tau) = \frac{1}{1 - \tau z} + \sqrt{\frac{z}{1 + \tau}} \frac{\arctan \left(\frac{2\sqrt{z + \tau z}\sqrt{1 - z\tau}}{2 - z - 2\tau z} \right)}{(1 - \tau z)^{3/2}} \tag{62}$$

and it satisfies

$$\left[\left(\frac{1}{z} - 3\tau - \frac{5}{2} \right) - z(1 + \tau) \frac{\partial}{\partial z} + \frac{(1 + \tau)(2 - \tau z)}{z} \frac{\partial}{\partial \tau} \right] \phi(\beta = 1/2, z, \tau) = \frac{1}{z} \tag{63}$$

with the initial conditions $\phi(\beta = 1/2, z, \tau) = 1 + (\tau + 1/2)z + \ldots$

One can see that in both cases of $\beta = 2, 1/2$, which correspond to classical groups, the Harer-Zagier functions remain expressed in terms of arctangents. This, however, is not the case in the general situation.

The case of $\beta = 3$. The Harer-Zagier function for $\beta = 3$ satisfies the differential equation

$$(1 + 8z^2\tau + 24z^2\tau^2 + 9z^3\tau - z\tau - 6z - 33z^2)\phi + (11z^2\tau - 18z^3 - 2z + 9z^4\tau)\frac{\partial\phi}{\partial z} \tag{64}$$

$$+(9z - 12z^2\tau + 12z^2\tau^3 - 9z\tau^2)\frac{\partial\phi}{\partial\tau} = 1 - 4z - 4z\tau$$

at particular values of z it becomes the hypergeometric equation and, hence, has no solutions expressed in elementary functions. So, presumably, what one is searching for is some clever deformation of the arctangent function from the previously described cases.

We observe that as we move further and further away from $\beta = 1$, the complexity of results increases. Further work is needed to clarify the situation.

A.3 Identities for free energy

It turns out that for $\beta \neq 1$ the Gaussian free energy has more structure than one could expect.

A.3.1 Definitions

Let us define the partition function without $\frac{1}{N!}$ factor. To avoid an ambiguity, let us denote all the quantities in this normalization with tildes.

The partition function for the Gaussian model we are considering is

$$\widetilde{Z}(N,\beta) = \int d\lambda_1 \dots d\lambda_N \prod_{i<j}(\lambda_i - \lambda_j)^{2\beta}\, e^{-\frac{1}{2g}\sum_i \lambda_i^2} = N! Z(N,\beta) \tag{65}$$

Instead of (49) we now have

$$\widetilde{Z}(N,\beta) = \sqrt{2\pi}^N \sqrt{g}^{\,\beta N^2 + (1-\beta)N} \prod_{k=1}^{N} \frac{\Gamma(1+\beta k)}{\Gamma(1+\beta)} \tag{66}$$

Defining the free energy as

$$\widetilde{F}(N,\beta) = \ln\widetilde{Z} \sim \sum_{k=1}^{N} \ln\Gamma\left(1 + \beta k\right), \tag{67}$$

where the equivalence means equality up to terms quadratic and linear in the matrix size N (they can be absorbed into redefinition of β and g).

A.3.2 Difference equation

Thus defined free energy satisfies a certain difference equation. Consider

$$\widetilde{G}(N,\beta) = \widetilde{F}(N,\beta) - \widetilde{F}(N-1,\beta) = \ln\Gamma\left(1 + \beta N\right) \tag{68}$$

then it is obvious that

$$\widetilde{G}(N,\beta) - \widetilde{G}\left(N - \frac{1}{\beta}, \beta\right) = \ln\left(\beta N\right) \tag{69}$$

which implies that

$$\frac{\partial}{\partial N}\widetilde{G}(N,\beta) - \frac{\partial}{\partial N}\widetilde{G}\left(N - \frac{1}{\beta}, \beta\right) = \frac{1}{N} \tag{70}$$

A.3.3 Exact relation between $\mathcal{F}_{SW}$ and $\widetilde{F}$

Comparison of (56) and (70) gives

$$\boxed{\mathcal{F}_{SW}(N,\beta) = \widetilde{F}\left(N - \frac{1}{\beta}, \beta\right) = \sum_{k}^{N-\frac{1}{\beta}} \ln\Gamma(1+\beta k),} \tag{71}$$

so the only peculiarity is in change of upper limit of summation. In the case of $\beta = 1$, it becomes $N - 1$ and acquires a clear physical meaning: division of partition function by $N!$ implies that the eigenvalues are indistinguishable bosons.

A.3.4 Direct comparison of series

However, it is still instructive not to appeal to this difference equation argument, but to act straightforwardly and look directly at the perturbative expansions at large N in order to see, what one can deduce from them. One way to obtain these expansions is to use the Euler-Maclaurin formula.

Euler-Maclaurin formula. We need this formula in the following form:

$$\frac{\partial}{\partial N} \sum_{k}^{N-1} f(k) = f(N) - \frac{1}{2} f'(N) + \frac{1}{12} f''(N) - \frac{1}{720} f''''(N) - \ldots = \sum_{m=0}^{\infty} \frac{B_m}{m!} \, \partial^m f(N), \tag{72}$$

where B_m are the Bernoulli numbers, $\sum \frac{B_m}{m!} t^m = \frac{t}{e^t - 1}$. The low limit in the sum is inessential, as long as it does not depend on N. In the following examples it is chosen to be $k = 0$.

The first examples are

$$
\begin{array}{llll}
f(k) = 1 & & \frac{\partial}{\partial N} N = 1 & \\
f(k) = k & & \frac{\partial}{\partial N} \frac{N(N-1)}{2} = N - \frac{1}{2} & \\
f(k) = k^2 & \frac{\partial}{\partial N} \frac{N(N-1)(2N-1)}{6} = N^2 - N + \frac{1}{6} & = N^2 - \frac{2N}{2} + \frac{2}{12} \\
f(k) = k^3 & \frac{\partial}{\partial N} \frac{N^2(N-1)^2}{4} = N^3 - \frac{3}{2} N^2 + \frac{1}{2} N & = N^3 - \frac{3N^2}{2} + \frac{6N}{12} + 0 \\
& \cdots &
\end{array}
$$

Different series as they are. Here all equalities are considered up to terms linear and constant in N or Λ. Summing up contributions from different genera, one gets

$$\frac{\partial}{\partial N} \mathcal{F}_{SW}\left(\frac{\Lambda}{g}\right) = \frac{1}{g}\beta\Lambda \ln\Lambda + \frac{\beta-1}{2} \ln\Lambda + g\frac{1-3\beta+\beta^2}{12\beta\Lambda} + g^2\frac{\beta-1}{24\beta\Lambda^2} + g^3\frac{-1+5\beta^2-\beta^4}{360\beta^3\Lambda^3} + \tag{73}$$

$$+ g^4\frac{1-\beta^3}{240\beta^3\Lambda^4} + g^5\frac{2-7\beta^2-7\beta^4+2\beta^6}{2520\beta^5\Lambda^5} + o\left(\frac{1}{\Lambda^5}\right)$$

Expanding F at various points one gets

$$\frac{\partial}{\partial N} \widetilde{F}\left(\frac{\Lambda}{g} - \frac{1}{\beta}\right) = \frac{1}{g}\beta\Lambda \ln\Lambda + \frac{\beta-1}{2} \ln\Lambda + g\frac{1-3\beta+\beta^2}{12\beta\Lambda} + g^2\frac{\beta-1}{24\beta\Lambda^2} + g^3\frac{-1+5\beta^2-\beta^4}{360\beta^3\Lambda^3} +$$

$$+ g^4\frac{1-\beta^3}{240\beta^3\Lambda^4} + g^5\frac{2-7\beta^2-7\beta^4+2\beta^6}{2520\beta^5\Lambda^5} + o\left(\frac{1}{\Lambda^5}\right)$$

$$\frac{\partial}{\partial N} \widetilde{F}\left(\frac{\Lambda}{g}\right) = \frac{1}{g}\beta\Lambda \ln\Lambda + \frac{1+\beta}{2} \ln\Lambda + g\frac{1+3\beta+\beta^2}{12\beta\Lambda} - g^2\frac{1+\beta}{24\beta\Lambda^2} - g^3\frac{1-5\beta^2+\beta^4}{360\beta^3\Lambda^3} + \tag{74}$$

$$+ g^4\frac{1+\beta^3}{240\beta^3\Lambda^4} + g^5\frac{2-7\beta^2-7\beta^4+2\beta^6}{2520\beta^5\Lambda^5} + o\left(\frac{1}{\Lambda^5}\right)$$

$$\frac{\partial}{\partial N} \widetilde{F}\left(\frac{\Lambda}{g} - 1\right) = \frac{1}{g}\beta\Lambda \ln\Lambda + \frac{1-\beta}{2} \ln\Lambda + g\frac{1-3\beta+\beta^2}{12\beta\Lambda} + g^2\frac{1-\beta}{24\beta\Lambda^2} - g^3\frac{1-5\beta^2+\beta^4}{360\beta^3\Lambda^3} +$$

$$+ g^4\frac{\beta^3-1}{240\beta^3\Lambda^4} + g^5\frac{2-7\beta^2-7\beta^4+2\beta^6}{2520\beta^5\Lambda^5} + o\left(\frac{1}{\Lambda^5}\right)$$

Interpretation. From the above series, one can easily guess the following relations

$$\boxed{\mathcal{F}_{SW}\left(\frac{\Lambda}{g}, \beta\right) = \widetilde{F}\left(\frac{\Lambda}{g} - \frac{1}{\beta}, \beta\right) = -\widetilde{F}\left(-\frac{\Lambda}{g} - 1, \beta\right) = -\widetilde{F}\left(\frac{\Lambda}{g}, -\beta\right),} \tag{75}$$

The first equality is expected to hold from our previous difference equation analysis.

It turns out that $\widetilde{F}$ with shifted arguments also satisfies the same difference equation. Indeed,

$$-\widetilde{F}(N, -\beta) + \widetilde{F}(N-1, -\beta) = -\ln\Gamma(1 - \beta N),$$
$$-\ln\Gamma(-\beta N) + \ln\Gamma(1 - \beta N) = \ln(-\beta N) \tag{76}$$

For $\widetilde{F}(-N-1, \beta)$ one has to assume that the lower limit of summation is less than $-N-1$ (which is rather weird)

$$-\widetilde{F}(-N-1, \beta) + \widetilde{F}(-N, \beta) = -\ln\Gamma(1 - \beta N) \tag{77}$$

Normally, shifting the expansion point for some function does not lead to series similar to the initial one, but produces something which looks completely different. The fact that this is not the case may be an indication that some not yet discovered mathematical structure is present here. Perhaps, it is a peculiar feature of the Gaussian potential or, may be such equalities have more general character. It is interesting to see which of these unexpected identities survive generalization to non-Gaussian eigenvalue models.

References

[1] N.Seiberg and E.Witten, Nucl.Phys., **B426** (1994) 19-52, hep-th/9408099; Nucl.Phys., **B431** (1994) 484-550, hep-th/9407087

[2] A.Klemm, W.Lerche, S.Theisen and S.Yankielowicz, Phys.Lett. **B344** (1995) 169-175, arXiv:hep-th/9411048;
P.Argyres and A.Shapere, Nucl.Phys. **B461** (1996) 437-459, hep-th/9509175

[3] A.Gorsky, I.Krichever, A.Marshakov, A.Mironov, A.Morozov, Phys.Lett., **B355** (1995) 466-477, hep-th/9505035;
R.Donagi and E.Witten, Nucl.Phys., **B460** (1996) 299-334, hep-th/9510101

[4] I.Krichever, hep-th/9205110; *Comm.Math.Phys.* **143** (1992) 415;
B.Dubrovin, *Nucl.Phys.* **B379** (1992) 627;
H.Itoyama and A.Morozov, *Nucl.Phys.* **B491** (1997) 529, hep-th/9512161;
A.Marshakov, *Seiberg-Witten Theory and Integrable Systems*, World Scientific 1999;
A.Gorsky and A.Mironov, hep-th/0011197

[5] E.Witten, Surv.Diff.Geom., **1** (1991) 243;
R.Dijkgraaf, E.Verlinde and H.Verlinde, Nucl.Phys., **B352** (1991) 59;
A.Marshakov, A.Mironov and A.Morozov, Phys. Lett. **B389** (1996) 43, arXiv:hep-th/9607109;
Mod.Phys.Lett. **A12** (1997) 773-787, arXiv:hep-th/9701014; Int.J.Mod.Phys. **A15** (2000) 1157-1206, arXiv:hep-th/9701123;
A.Mironov and A.Morozov, Phys.Lett., **B424** (1998) 48-52, hep-th/9712177
A.Veselov, hep-th/9902142;
H.W. Braden, A.Marshakov, A.Mironov and A.Morozov, Phys.Lett., **B448** (1999) 195, hep-th/9812078

[6] A.Mironov and A.Morozov, JHEP **04** (2010) 040, arXiv:0910.5670; J.Phys. **A43** (2010) 195401, arXiv:0911.2396;
A.Popolitov, arXiv:1001.1407;
Wei He, Yan-Gang Miao, Phys.Rev. **D82** (2010) 025020, arXiv:1006.1214;
K.Maruyoshi and M.Taki, arXiv:1006.4505;
F.Fucito, J.F.Morales, R.Poghossian and D. Ricci Pacifici, arXiv:1103.4495;
Y.Zenkevich, arXiv:1103.4843

[7] A.Marshakov, A.Mironov and A.Morozov, arXiv:1011.4491

[8] G.Moore, N.Nekrasov, S.Shatashvili, Nucl.Phys. **B534** (1998) 549-611, hep-th/9711108; hep-th/9801061;
A.Losev, N.Nekrasov and S.Shatashvili, Commun.Math.Phys. **209** (2000) 97-121, hep-th/9712241; ibid. 77-95, hep-th/9803265

[9] N.Nekrasov and S.Shatashvili, arXiv:0908.4052

[10] A.Mironov, A.Morozov, Sh.Shakirov, JHEP **02** (2010) 030, arXiv:0911.5721

[11] A.Mironov, A.Morozov and Sh.Shakirov, arXiv:1011.5629

[12] L.Alday, D.Gaiotto and Y.Tachikawa, Lett.Math.Phys. **91** (2010) 167-197, arXiv:0906.3219;
N.Wyllard, JHEP **0911** (2009) 002, arXiv:0907.2189;
A.Mironov and A.Morozov, Phys.Lett. **B680** (2009) 188-194, arXiv:0908.2190; Nucl.Phys. **B825** (2009) 1-37, arXiv:0908.2569

[13] L.Hadasz, Z.Jaskolski and P.Suchanek, arXiv:0911.2353; arXiv:1004.1841;
V.Fateev and I.Litvinov, JHEP **1002** (2010) 014, arXiv:0912.0504;
V.Alba, V.Fateev, A.Litvinov and G.Tarnopolsky, arXiv:1012.1312;
A.Belavin and V.Belavin, arXiv:1102.0343

[14] R.Dijkgraaf and C.Vafa, arXiv:0909.2453;
H.Itoyama, K.Maruyoshi and T.Oota, Prog.Theor.Phys. **123** (2010) 957-987, arXiv:0911.4244;
T.Eguchi and K.Maruyoshi, arXiv:0911.4797; arXiv:1006.0828;
R.Schiappa and N.Wyllard, arXiv:0911.5337

[15] H.Itoyama and T.Oota, arXiv:1003.2929;
A.Mironov, A.Morozov and And.Morozov, arXiv:1003.5752;
A.Mironov, A.Morozov and Sh.Shakirov, JHEP **02** (2011) 067, arXiv:1012.3137

[16] R.Dijkgraaf and C.Vafa, Nucl.Phys. **B644** (2002) 3, hep-th/0206255; Nucl.Phys. **B644** (2002) 21, hep-th/0207106; hep-th/0208048

[17] L.Chekhov and A.Mironov, Phys.Lett. **B552** (2003) 293, hep-th/0209085;
A.Klemm, M.Marino and S.Theisen, JHEP 0303 (2003) 051, hep-th/0211216;
V.Kazakov and A.Marshakov, J.Phys. **A36** (2003) 3107, hep-th/0211236;
H.Itoyama and A.Morozov, Nucl.Phys.B657:53-78,2003, hep-th/0211245; Phys.Lett. B555 (2003) 287-295, hep-th/0211259; Prog.Theor.Phys. 109 (2003) 433-463, hep-th/0212032; Int.J.Mod.Phys. A18 (2003) 5889-5906, hep-th/0301136;
L.Chekhov, A.Marshakov, A.Mironov and D.Vasiliev, Phys.Lett. **B562** (2003) 323, hep-th/0301071; Proc. Steklov Inst.Math. **251** (2005) 254, hep-th/0506075;
A.Alexandrov, A.Mironov and A.Morozov, Int.J.Mod.Phys. **A21** (2006) 2481-2518, hep-th/0412099; Fortsch.Phys. **53** (2005) 512-521, hep-th/0412205;
A.Mironov, Theor.Math.Phys. **146** (2006) 63-72, hep-th/0506158

[18] A.Alexandrov, A.Mironov and A.Morozov, Int.J.Mod.Phys. **A19** (2004) 4127, hep-th/0310113

[19] B.Eynard and N.Orantin, arXiv:math-ph/0702045

[20] A.Alexandrov, A.Mironov and A.Morozov, Teor.Mat.Fiz. **150** (2007) 179-192, hep-th/0605171; Physica **D235** (2007) 126-167, hep-th/0608228; JHEP **12** (2009) 053, arXiv:0906.3305;
A.Alexandrov, A.Mironov, A.Morozov, P.Putrov, Int.J.Mod.Phys. **A24** (2009) 4939-4998, arXiv:0811.2825;
B.Eynard, JHEP **0411** (2004) 031, hep-th/0407261;
L.Chekhov and B.Eynard, JHEP **0603** (2006) 014, hep-th/0504116; JHEP **0612** (2006) 026, math-ph/0604014;
N.Orantin, arXiv:0808.0635;
I.Kostov and N.Orantin, arXiv:1006.2028;
L.Chekhov, B.Eynard and O.Marchal, arXiv:1009.6007

[21] A.Mironov, A.Morozov, Sh.Shakirov, Int.J.Mod.Phys. **A25** (2010) 3173-3207, arXiv:1001.0563

[22] A.Gerasimov, A.Marshakov, A.Mironov, A.Morozov and A.Orlov, Nucl.Phys. **B357** (1991) 565-618

[23] A.A. Migdal, Phys.Rep. **102** (1983) 199;
J. Ambjørn, J. Jurkiewicz and Yu. Makeenko, Phys.Lett. **B251** (1990) 517;
F. David, Mod.Phys.Lett. **A5** (1990) 1019;
A. Mironov and A. Morozov, Phys.Lett. **B252** (1990) 47-52
J. Ambjørn and Yu. Makeenko, Mod.Phys.Lett. **A5** (1990) 1753;
H. Itoyama and Y. Matsuo, Phys.Lett. **255B** (1991) 202

[24] A.Marshakov, A.Mironov, and A.Morozov, Phys.Lett. **B265** (1991) 99
S.Kharchev, A.Marshakov, A.Mironov, A.Morozov and S.Pakuliak, Nucl.Phys. **B404** (1993) 17-750, arXiv:hep-th/9208044;
H.Awata, Y.Matsuo, S.Odake and J.Shiraishi, Phys. Lett. **B347** (1995) 49, hep-th/9411053; Soryushiron Kenkyu **91** (1995) A69-A75, hep-th/9503028

[25] A.Morozov and Sh.Shakirov, arXiv:1004.2917

[26] J.Harer and D.Zagier, Invent.Math. **85** (1986) 457-485;
S.K.Lando and A.K.Zvonkin, *Embedded graphs*, Max-Plank-Institut für Mathematik, Preprint Series 2001 (**63**)

[27] A.Morozov and Sh.Shakirov, JHEP **0912** (2009) 003 arXiv:0906.003; arXiv:1007.4100